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Membrane solitons in eight-dimensional hyper-Kähler backgrounds

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Abstract: We derive the BPS equations satisfied by lump solitons in $(2+1)$-dimensional sigma models with toric 8-dimensional hyper-Kähler (HK_8) target spaces and check they preserve 1/2 of the supersymmetry. We show how these solitons are realised in M theory as M2-branes wrapping holomorphic 2-cycles in the $\mathbb{E}^{1,2} \times HK_8$ background. Using the κ-symmetry of a probe M2-brane in this background we determine the supersymmetry they preserve, and note that there is a discrepancy in the fraction of supersymmetry preserved by these solitons as viewed from the low energy effective sigma model description of the M2-brane dynamics or the full M theory. Toric HK_8 manifolds are dual to a Hanany-Witten setup of D3-branes suspended between 5-branes. In this picture the lumps correspond to vortices of the three dimensional $\mathcal{N}=3$ or $\mathcal{N}=4$ theory.

Keywords: Sigma Models, Solitons Monopoles and Instantons, M-Theory.

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Contents

1. Introduction

In the absence of background field strengths, the dynamics of branes with vanishing worldvolume gauge fields embedded in a $(D+1)$-dimensional spacetime with coordinates X^M and metric G_{MN} is governed by the Nambu-Goto action

$$S = -\int d^{p+1}\sigma \sqrt{-\det g}\,. \tag{1.1}$$

Here σ^μ $(\mu = 0, \ldots, p)$ denote the worldvolume coordinates, g is the pull-back of the spacetime metric onto the worldvolume and the brane tension has been set to 1. It is often convenient to fix the worldvolume diffeomorphism invariance by setting

$$X^\mu(\sigma) = \sigma^\mu\,. \tag{1.2}$$

We take the metric G_{MN} to be the direct product of the flat metric η on the worldvolume and a metric G_{AB} $(A, B = p+1, \ldots, D)$ on the riemannian space transverse to the brane. The induced metric becomes

$$g_{\mu\nu} = \eta_{\mu\nu} + \partial_\mu X^A \partial_\nu X^B G_{AB}\,. \tag{1.3}$$

Ignoring a constant term related to the mass of the brane, the lagrangian $\mathcal{L}$ in (1.1) can be expanded to give the sigma model lagrangian density:

$$\mathcal{L} \approx -\frac{1}{2}\eta^{\mu\nu}\partial_\mu X^A \partial_\nu X^B G_{AB} + \mathcal{O}\left((\partial X)^4\right). \tag{1.4}$$

JHEP03(2004)051

We therefore obtain the result that the low energy 'non-relativistic' fluctuations of a p-brane are governed by a $(p+1)$-dimensional sigma model which describes the embedding of the brane in its transverse space, where the sigma model fields X^A are these transverse coordinates.[1]

In [2], the lumps, kinks and Q-kinks of sigma-models with a four dimensional hyper-Kähler (HK_4) target space were found to be realised in M theory as intersecting M2-branes in a KK-monopole background and dimensional reductions of this configuration. In that case the sigma model describes the fluctuations of one of the two M2-branes and the other M2-brane appears in the worldvolume theory of the former as a lump-membrane. This background was chosen because a consistent supersymmetric sigma model can have at most 8 supersymmetries. The KK-monopole halves the 32 supersymmetries of M theory and then the M2-brane on which the sigma model is defined, and which describes its vacuum, halves this again to 8. The approach was extended in [3] and [4], where new static and stationary sigma model solitons were found and their corresponding realisations in M theory studied.

The aim of this paper is to study another class of sigma models with relevance in M theory, namely $\mathcal{N}=4$ $(2+1)$-dimensional sigma models with an eight dimensional hyper-Kähler (HK_8) target space. We start in section 2 by presenting a brief introduction to toric HK_8 manifolds. We use an explicit form of the metric for these manifolds which includes a parameter λ, such that for $\lambda=0$ the HK_8 manifold degenerates into the direct product $HK_4\times HK_4$. In section 3 we study a $(2+1)$-dimensional sigma model with an HK_8 manifold as a target space. We describe a consistent truncation that can be performed on such a model that yields a 4-dimensional Kähler manifold as the "new" target space. We obtain the first order Bogomol'nyi equations satisfied by lumps by using the standard procedure of writing the energy density as a sum of squares plus a topological term. We also check the fraction of supersymmetry preserved by the lumps. In section 4 we proceed to study the M theory picture. In order to do this we chose to examine a probe M2-brane in a $\mathbb{E}^{1,2}\times HK_8$ background which is a solution to 11-dimensional supergravity. First of all we find that for the generic case $\lambda\neq 0$, the vacuum of the theory preserves only 6 supersymemtries, and therefore corresponds to an $\mathcal{N}=3$ theory in 2+1 dimensions. This reduction in the number of supersymmetries when compared to the sigma model was argued in [6] to be due to the higher order fermionic interactions not captured by the sigma model.[2] We then use the BPS equations derived from the sigma model to determine the fraction of supersymmetry preserved by the membrane lumps in M theory. The results of the paper are summarised in table 1. We note the discrepancy in the supersymmetry preserved by the lumps as seen from the sigma model analysis and from the probe M2-brane perspective.

It is natural to ask why we expect the BPS equations derived from the sigma model to be valid in M theory. Lumps on the membrane correspond to intersections with other membranes wrapping Kähler calibrated 2-cycles. The full M theory equations describing the lump will be those that determine such a calibrated surface. A Kähler calibration

[1] For a more detailed treatment see [1].

[2] We note that an $\mathcal{N}=3$ sigma model is automatically $\mathcal{N}=4$ [7].

JHEP03(2004)051

		$\lambda = 0$	$\lambda \neq 0$
Sigma model	Vacuum	8	8
	Lump - 2 active fields	4	4
	Lump - 4 active fields	4	4
M theory	Vacuum	8	6
	Lump - 2 active fields	4	no solution
	Lump - 4 active fields	2	2

Table 1: We exhibit the supersymmetries of the vacuum theories and those preserved by solutions to the lump equations in both the sigma model and the full M theory, for the generic HK_8 manifold ($\lambda \neq 0$), and for the degenerate case when the HK_8 manifold is the direct product $\mathrm{HK}_4 \times \mathrm{HK}_4$ ($\lambda = 0$).

however, is fully determined by linear first order differential equations, and we expect to be able to trust the sigma model to this order. For other calibrations, the sigma model agrees only to linearized order [4]. In section 5 we conclude with a summary of these results and some comments on a Hanany-Witten dual version of the configurations considered here whose low energy effective theory is a gauge theory in (2+1) dimensions with either $\mathcal{N} = 4$ or $\mathcal{N} = 3$ depending on whether the HK_8 manifold a direct product $\mathrm{HK}_4 \times \mathrm{HK}_4$ or not.

2. HK_8 manifolds

In this paper we will consider a sigma model with a toric eight dimensional hyper-Kähler target space. These manifolds have the metric

$$ds^2 = U_{\alpha\beta} d\boldsymbol{\mathcal{X}}^\alpha \cdot d\boldsymbol{\mathcal{X}}^\beta + U^{\alpha\beta}(d\varphi_\alpha + \mathcal{A}_\alpha)(d\varphi_\beta + \mathcal{A}_\beta)\,, \tag{2.1}$$

where $\alpha, \beta = 1, 2$, $\boldsymbol{\mathcal{X}}^\alpha = (\boldsymbol{X}, \boldsymbol{Y})$, $\varphi_\alpha = (\varphi, \psi)$, $U_{\alpha\beta}$ is a matrix of funxtions of $\boldsymbol{X}$ and $\boldsymbol{Y}$ and $\mathcal{A}_\alpha = (A, B)$ is a doublet of one forms. This metric admits a triholomorphic T^2 action generated by the two Killing vector fields $\partial/\partial\varphi$ and $\partial/\partial\psi$, and admits the triplet of Kähler two-forms

$$\boldsymbol{\Omega} = (d\varphi_\alpha + A_\alpha) d\boldsymbol{\mathcal{X}}^\alpha - \frac{1}{2} U_{\alpha\beta} d\boldsymbol{\mathcal{X}}^\alpha \times d\boldsymbol{\mathcal{X}}^\beta\,, \tag{2.2}$$

where the wedge product of forms is understood. Without loss of generality, we take

$$U_{\alpha\beta} = \begin{pmatrix} U & W \\ W & V \end{pmatrix}, \tag{2.3}$$

where U, V and W are functions of the six coordinates $(\boldsymbol{X}, \boldsymbol{Y})$. The inverse matrix $(U^{-1})_{\alpha\beta}$ is given by

$$(U^{-1})_{\alpha\beta} = (UV - W^2)^{-1} \begin{pmatrix} V & -W \\ -W & U \end{pmatrix}. \tag{2.4}$$

The 1-forms A and B satisfy certain conditions analogous to the usual $\nabla \times A = \pm\nabla U$ of HK_4 manifolds which ensure that (2.1) is Ricci-flat. These conditions will play no part in what follows, so we refer the reader to [8, 9] for details. Therefore the most general form of the metric for a toric HK_8 manifold is

$$\begin{aligned} ds^2(\mathrm{HK}_8) &= U d\boldsymbol{X}\cdot d\boldsymbol{X} + V d\boldsymbol{Y}\cdot d\boldsymbol{Y} + 2W d\boldsymbol{X}\cdot d\boldsymbol{Y} + (UV-W^2)^{-1}\times \\ &\quad\times\big(V(d\varphi+A)^2 + U(d\psi+B)^2 - 2W(d\varphi+A)(d\psi+B)\big)\,. \end{aligned} \tag{2.5}$$

For future use we also explicitly write down

$$\Omega^{(3)} = \mathcal{D}\varphi\wedge dX_3 + \Delta\psi\wedge dY_3 - U dX_1\wedge dX_2 - V dY_1\wedge dY_2 - W(dX_1\wedge dY_2 + dY_1\wedge dX_2)\,, \tag{2.6}$$

where we have introduced the “covariant” derivatives

$$\mathcal{D}\varphi = d\varphi + A\,,\qquad \Delta\psi = d\psi + B\,. \tag{2.7}$$

Generically, when $W\neq 0$, the manifold with metric (2.5) will have holonomy Sp_2. The 32 component $\mathrm{SO}(10,1)$ Majorana spinor of eleven dimensions will give 6 Sp_2 singlets when decomposed in terms of its Sp_2 subgroup, as we shall check explicitly in section 4.1. However when $W=0$ the HK_8 manifold becomes the direct product $\mathrm{HK}_4\times\mathrm{HK}_4$, and the holonomy is accordingly enhanced to $Sp_1\times Sp_1$. When this is the case, the eleven dimensional spinor decomposes to give 8 singlets of this subgroup. It will be convenient in what follows to set the function W to be a non-vanishing constant, so from now on we set it to λ. This ensures that the preceeding discussion applies, and allows us to choose a gauge in which the different $\mathrm{U}(1)$ connections A_i can be set to zero, such that the covariant derivatives become the usual partial derivatives.

3. The sigma model approach

3.1 The model and the BPS equations

We wish to study the effective $(2+1)$-dimensional sigma model describing the fluctuations of an M2-brane. The lagrangian is

$$\mathcal{L} = -\frac{1}{2}\eta^{ij}\partial_i\Phi^A\partial_j\Phi^B G_{AB}\,, \tag{3.1}$$

where $\Phi^A = (\boldsymbol{X},\varphi,\boldsymbol{Y},\psi)$ $(A,B=1,\ldots,8)$ and $i,j=0,1,2$ and G_{AB} is given by (2.5). The metric (2.5) allows two $\mathrm{SO}(3)$ actions, each acting on one of the sets $(\boldsymbol{X},\varphi)$ and $(\boldsymbol{Y},\psi)$. In both cases the angular variables transform in the trivial representation, whereas $\boldsymbol{X}$ and $\boldsymbol{Y}$ transform in the **3**. In the lagrangian (3.1) however, when $\lambda\neq 0$ the term from the metric involving $2\lambda d\boldsymbol{X}\cdot d\boldsymbol{Y}$ breaks this to the diagonal $\mathrm{SO}(3)$. The energy density associated to the lagrangian (3.1) is

$$\begin{aligned} \mathcal{E} &= U|\dot{\boldsymbol{X}}|^2 + U|\boldsymbol{\nabla}\boldsymbol{X}|^2 + V|\dot{\boldsymbol{Y}}|^2 + V|\boldsymbol{\nabla}\boldsymbol{Y}|^2 + 2\lambda|\dot{\boldsymbol{X}}\cdot\dot{\boldsymbol{Y}}| + 2\lambda|\boldsymbol{\nabla}\boldsymbol{X}\cdot\boldsymbol{\nabla}\boldsymbol{Y}| + \\ &\quad + (UV-\lambda^2)^{-1}[V(\mathcal{D}_t\varphi)^2 + V|\boldsymbol{\mathcal{D}}\varphi|^2 + U(\Delta_t\psi)^2 + U|\boldsymbol{\Delta}\psi|^2 - \\ &\quad - 2\lambda(\mathcal{D}_t\varphi)(\Delta_t\psi) - 2\lambda(\boldsymbol{\mathcal{D}}\varphi\cdot\boldsymbol{\Delta}\psi)]\,, \end{aligned} \tag{3.2}$$

JHEP03(2004)051

where $\boldsymbol{\nabla} = (\partial_1, \partial_2)$ and the covariant derivaties defined in (2.7) have been used. We will now choose a two-centered form for the harmonic functions $U(\boldsymbol{X})$ and $V(\boldsymbol{Y})$

$$U(\boldsymbol{X}) = \text{const.} + \frac{1}{2}\left(\frac{1}{|\boldsymbol{X} + a\hat{X}_3|} + \frac{1}{|\boldsymbol{X} - a\hat{X}_3|}\right), \tag{3.3}$$

and a similar expression for $V(\boldsymbol{Y})$ with two centres at $\boldsymbol{Y} = \pm b\hat{Y}_3$, where the hats denote unit vectors. For this choice of functions, and any other functions with colinear centers, the metric (2.5) (with $W = \lambda$) has two additional Killing vector fields which generate rotations around the $\hat{X}_3$ and $\hat{Y}_3$ axes. This breaks the SO(3) isometry in the $\boldsymbol{X}$- and $\boldsymbol{Y}$-planes to SO(2). The fields will decompose into representations of these subgroups, and in particular $\mathbf{3} \to \mathbf{2} \oplus \mathbf{1}$. We will now study the truncated model in which we only keep the SO(2) singlets. Such truncations are always consistent, that is, solutions to the truncated models will automatically be solutions to the full model with the other fields set to zero. This is easy to see by considering that every term in a lagrangian must transform trivially under a symmetry tranformation. Therefore any term which couples a singlet and non-singlet fields must contain at least two of the latter.

The truncated model has non-trivial fields φ, ψ, X_3, Y_3 and from now on we will write the last two as X and Y for simplicity. This model has a $\mathrm{U}(1)_L \times \mathrm{U}(1)_R$ symmetry, which is again further broken to the diagonal U(1) when $\lambda \neq 0$. It constitutes a truncation to a Kähler model with metric

$$ds^2 = U dX^2 + V dY^2 + 2\lambda dX dY + (UV - \lambda^2)^{-1}\left(V d\varphi^2 + U d\psi^2 - 2\lambda d\varphi d\psi\right), \tag{3.4}$$

and Kähler two form

$$\Omega = d\varphi \wedge dX + d\psi \wedge dY\,. \tag{3.5}$$

The energy density becomes

$$\begin{aligned}\mathcal{E} = {} & U\left((\partial_1 X)^2 + (\partial_2 X)^2\right) + V\left((\partial_1 Y)^2 + (\partial_2 Y)^2\right) + 2\lambda\left(\partial_1 X \partial_1 Y + \partial_2 X \partial_2 Y\right) + \\ & + (UV - \lambda^2)^{-1}[V\left((\partial_1\varphi)^2 + (\partial_2\varphi)^2\right) + U\left((\partial_1\psi)^2 + (\partial_2\psi)^2\right) - \\ & \qquad - 2\lambda\left(\partial_1\varphi\partial_1\psi + \partial_2\varphi\partial_2\psi\right)]\,. \end{aligned} \tag{3.6}$$

Remarkably, it is now possible to write (3.6) as

$$\begin{aligned}\mathcal{E} = {} & U\left(\partial_1 X + U^{-1}\partial_2\varphi + \frac{\lambda}{U}\partial_1 Y\right)^2 + U\left(\partial_2 X - U^{-1}\partial_1\varphi + \frac{\lambda}{U}\partial_2 Y\right)^2 + \\ & + V\left(\sqrt{\frac{UV - \lambda^2}{UV}}\partial_1 Y + V^{-1}\sqrt{\frac{UV}{UV - \lambda^2}}\partial_2\psi - \frac{\lambda}{\sqrt{UV(UV - \lambda^2)}}\partial_2\varphi\right)^2 + \\ & + V\left(\sqrt{\frac{UV - \lambda^2}{UV}}\partial_2 Y - V^{-1}\sqrt{\frac{UV}{UV - \lambda^2}}\partial_1\psi + \frac{\lambda}{\sqrt{UV(UV - \lambda^2)}}\partial_1\varphi\right)^2 + \\ & + 2(\partial_2 X \partial_1\varphi - \partial_1 X \partial_2\varphi + \partial_2 Y \partial_1\psi - \partial_1 Y \partial_2\psi)\,, \end{aligned} \tag{3.7}$$

that is, as a sum of squares with positive coefficients plus a topological term given by the pull-back of the Kähler two form (3.5) onto the worldvolume. A very similar form corresponding to interchanging X and Y can also be written down, though as will become evident underneath, it will lead to the same soliton equations. We therefore expect the existence of solitons, whose explicit solutions are given by the vanishing of the positive semi-definite terms and which are supported by a topological charge, which in this case is the pull-back of the Kähler 2-form (2.6) of the truncated model. The linear BPS equations may be written as

$$\begin{aligned}
\partial_1 X &= -\frac{V}{UV-\lambda^2}\,\partial_2\varphi + \frac{\lambda}{UV-\lambda^2}\,\partial_2\psi \\
\partial_2 X &= \frac{V}{UV-\lambda^2}\,\partial_1\varphi - \frac{\lambda}{UV-\lambda^2}\,\partial_1\psi \\
\partial_1 Y &= -\frac{U}{UV-\lambda^2}\,\partial_2\psi + \frac{\lambda}{UV-\lambda^2}\,\partial_2\varphi \\
\partial_2 Y &= \frac{U}{UV-\lambda^2}\,\partial_1\psi - \frac{\lambda}{UV-\lambda^2}\,\partial_1\varphi\,.
\end{aligned} \tag{3.8}$$

Any solution to these equations will correspond to a two-dimensional, surface embedded in four dimensions and will be a solution to the full eight dimensional hyper-Kähler sigma model.

We also note that when $\lambda = 0$ we obtain two sets of the usual lump equations as written in [3]:

$$\begin{aligned}
\partial_1 X &= -U^{-1}\partial_2\varphi \qquad & \partial_2 X &= U^{-1}\partial_1\varphi \\
\partial_1 Y &= -V^{-1}\partial_2\psi \qquad & \partial_2 Y &= V^{-1}\partial_1\psi\,.
\end{aligned} \tag{3.9}$$

It is reassuring to check that the 2-surface described parametrically by (3.8) is calibrated, that is to say, that its volume form $\mathcal{V}$ is equal to the Kähler 2-form Ω, given by (3.5). To evaluate $\mathcal{V}$ on the 2-surface we simply need to pull-back the metric (3.4), which we denote by G_{IJ}, onto the 2-surface using the lump equations (3.8), to obtain the induced metric γ_{ab}:

$$\gamma_{ab} = \frac{\partial X^I}{\partial\sigma^a}\frac{\partial X^J}{\partial\sigma^b}G_{IJ}\,, \tag{3.10}$$

where $X^I = (X, Y, \varphi, \psi)$. Performing this calculation we find that:

$$\begin{aligned}
\gamma_{11} &= \gamma_{22} = |\Omega| \\
\gamma_{12} &= \gamma_{12} = 0\,,
\end{aligned} \tag{3.11}$$

showing that

$$-\det(\gamma) = |\Omega|^2\,. \tag{3.12}$$

3.2 Sigma model supersymmetry

As stated in the introduction supersymmetric sigma models have at most eight supercharges. Therefore all supersymmetric sigma models can be obtained from a 'maximal' $\mathcal{N} = 1$ model in $5+1$ dimensions via dimensional reduction. We regard the $\mathcal{N} = 4$ $(2+1)$-dimensional model (3.1) we are considering as being obtained from the maximal one by performing three trivial dimensional reductions.

JHEP03(2004)051

The conditions for a bosonic solution to a sigma model to be supersymmetric can be derived by requiring that the supersymmetry variations of the fermions vanish. For the target spaces considered here, the condition was derived in [10] to be

$$[\gamma^m \boldsymbol{\tau} \cdot \partial_m \boldsymbol{X}^\alpha + iU^{\alpha\beta}\gamma^m \mathcal{D}_m \varphi_\beta]\epsilon = 0\,, \tag{3.13}$$

where the γ^m are $D = 6$ Dirac matrices and the spinor ϵ satisfies the chirality constraint

$$\gamma^{012345}\epsilon = \epsilon\,, \tag{3.14}$$

and therefore has, a priori, 8 independent real components. The number of solutions to (3.13) will be the number of unbroken supersymmetries. Substituting the explicit form $U^{\alpha\beta}$ which we are considering into (3.13) we obtain the pair of equations

$$\begin{aligned}
\left[\tau^3(\gamma^1\partial_1 X + \gamma^2\partial_2 X) + \frac{iV}{UV-\lambda^2}(\gamma^1\partial_1\varphi + \gamma^2\partial_2\varphi) - \frac{i\lambda}{UV-\lambda^2}(\gamma^1\partial_1\psi + \gamma^2\partial_2\psi)\right]\epsilon &= 0\\
\left[\tau^3(\gamma^1\partial_1 Y + \gamma^2\partial_2 Y) + \frac{iU}{UV-\lambda^2}(\gamma^1\partial_1\psi + \gamma^2\partial_2\psi) - \frac{i\lambda}{UV-\lambda^2}(\gamma^1\partial_1\varphi + \gamma^2\partial_2\varphi)\right]\epsilon &= 0\,.
\end{aligned} \tag{3.15}$$

Now, substituting the lump equations (3.8) into (3.15) implies that (3.15) is satisfied provided the projector

$$i\tau^3\gamma^{12}\epsilon = \epsilon \tag{3.16}$$

is imposed on the spinor. This is the usual lump projector: it is traceless and squares to the identity and as it commutes with γ^{012345}, it halves the number of components of ϵ. There are therefore 4 solutions. We note here that solutions to (3.9) in the $\lambda = 0$ case also preserve the same amount of supersymmetry.

4. M theory supersymmetry

In this paper we consider an M2-brane as a probe brane in the spacetime

$$ds^2_{11d} = ds^2(\mathbb{E}^{(1,2)}) + ds^2(\mathrm{HK}_8)\,, \tag{4.1}$$

which is a Ricci flat solution of eleven dimensional supergravity, where $ds^2(\mathrm{HK}_8)$ is the metric (2.5) with $W = \lambda$.

Let us now consider a (bosonic) probe brane in this background. The global fermionic symmetries of the background will act on the membrane fields, and in general a solution will not be left invariant by these transformations. The condition for the brane to be supersymmetric is that there should exist a κ-symmetry transformation that combines with the global fermionic symmetries to leave the fields invariant [11, 12, 13].

In this approach the brane does not backreact. It was noted in [13] that considering backreacting branes does not impose any additional constraints on supersymmetry, that is, that a spinor will satisfy the Killing spinor equation if and only if it generates an appropiate κ-symmetry transformation on the worldvolume, as described above.

JHEP03(2004)051

4.1 Covariantly constant spinors in toric HK$_8$ manifolds

Before proceeding we need to determine the conditions imposed on the covariantly constant spinors of the metric (4.1). As explained in the appendix, where we also describe the gamma matrix conventions we use, we write an SO(10, 1) Majorana spinor ϵ as

$$\epsilon = \chi(\sigma^i) \otimes \eta(X^A)\,, \tag{4.2}$$

where χ is an anticommuting, SO(2, 1) spinor, and η is a commuting, SO(8) spinor. The condition that ϵ should be covariantly constant with respect to the metric (4.1) implies that η should be covariantly constant on the HK$_8$ manifold [9, 14]. We introduce a vielbein

$$\begin{aligned}
e^{X_i} &= U^{\frac{1}{2}}(dX^i + \lambda U^{-1}\delta_{i\alpha}dY^\alpha)\\
e^{\varphi} &= \left(\frac{V}{K}\right)^{\frac{1}{2}}\left[(d\varphi + A_i dX^i) - \lambda V^{-1}(d\psi + B_\alpha dY^\alpha)\right]\\
e^{Y_\alpha} &= \left(\frac{K}{U}\right)^{\frac{1}{2}} dY^\alpha\\
e^{\psi} &= V^{-\frac{1}{2}}(d\psi + B_\alpha dY^\alpha)\,,
\end{aligned} \tag{4.3}$$

where $i = 1, 2, 3$ labels the X coordinates, $\alpha = 1, 2, 3$ labels the Y coordinates and $K = UV - \lambda^2$. Killing spinors of (2.5) will be independent of the the coordinates φ and ψ [9]. This gives us the following conditions on the Killing spinors:

$$\begin{aligned}
\frac{1}{\sqrt{UV}}\Big[K\,\tilde\Gamma_{\varphi X_1X_2X_3} + \lambda\sqrt{UV}\,\tilde\Gamma_{X_1Y_1} -&\\
-\lambda\,\tilde\Gamma_{X_1\varphi}\Big(\sqrt{K}\,\tilde\Gamma_{X_3Y_2} + \lambda\,\tilde\Gamma_{Y_2Y_3} - \sqrt{K}\,\tilde\Gamma_{X_2Y_3}\Big)\Big]\eta &= \eta\\
\left[\sqrt{\frac{UV}{K}}\tilde\Gamma_{\psi Y_1Y_2Y_3} + \frac{\lambda}{\sqrt{K}}\tilde\Gamma_{\psi\varphi}\right]\eta &= \eta\,.
\end{aligned} \tag{4.4}$$

The explicit form of the Killing spinor can be determined by looking at the other components of the covariantly constant spinor equation, and we do so here for completeness. For $\lambda = 0$ the Killing spinor is in fact constant. For $\lambda \neq 0$ we obtain from $\nabla_{X_i}\eta = 0$ the condition

$$\eta = \exp\left[-2\tan^{-1}\left(\frac{\sqrt{K}}{\lambda}\right)\left(\tilde\Gamma_{X_1Y_1} + \tilde\Gamma_{X_2Y_2} + \tilde\Gamma_{X_3Y_3}\right)\right]\eta_0\,, \tag{4.5}$$

whereas from the covariant derivative $\nabla_{Y_\alpha}\eta = 0$ we obtain

$$\eta = \exp\left[-2\tan^{-1}\left(\frac{\sqrt{K}}{\lambda}\right)\tilde\Gamma_{\varphi\psi}\right]\eta_0\,. \tag{4.6}$$

These are consistent if and only if

$$\left(\tilde\Gamma_{X_1Y_1} + \tilde\Gamma_{X_2Y_2} + \tilde\Gamma_{X_3Y_3} - \tilde\Gamma_{\varphi\psi}\right)\eta_0 = 0\,. \tag{4.7}$$

This condition must be imposed on η_0 apart from those arising from (4.4).

JHEP03(2004)051

We can now determine the number of independent components of the Killing spinor preserved by (4.4). First of all we note that when $\lambda = 0$ (and $K = UV$) these simply yield $\tilde{\Gamma}_{\psi Y_1 Y_2 Y_3}\eta = \tilde{\Gamma}_{\varphi X_1 X_2 X_3}\eta = \eta$, which commute and therefore preserve 4 of the sixteen supersymmetries in 8 dimensions. To analyse (4.4) for $\lambda \neq 0$ we can choose a representation for the SO(8) Dirac matrices as explained in the appendix. It is then possible to check that the conditions (4.4) preserve three independent components of η in the generic $\lambda \neq 0$ case, as expected for a manifold of Sp_2 holonomy. We label these three independent solutions $\eta_{(i)}$ where $i = 1, 2, 3$. A general covariantly constant spinor of the metric (4.1) can therefore be written as

$$\epsilon = \sum_i \sum_s a_{si}(\chi_s \otimes \eta_{(i)}) \,, \tag{4.8}$$

where $s = +, -$, $\chi_+ = \binom{1}{0}$, $\chi_- = \binom{0}{1}$ and a_{si} are six arbitrary constants.

4.2 Supersymmetry analysis

As described at the beginning of the section and explained in [11, 12, 13], the M theory condition for a bosonic M2-brane solution to be supersymmetric is that there exists a worldvolume κ-symmetry transformation of the form

$$\Gamma_\kappa \epsilon = \epsilon \,, \tag{4.9}$$

where ϵ is a covariantly constant spinor of the background normalised such that $\epsilon^T \epsilon = 1$, and where

$$\Gamma_\kappa = \frac{1}{6\sqrt{-g}} \epsilon^{ijk} \partial_i X^M \partial_j X^N \partial_k X^P \Gamma_{MNP} \,. \tag{4.10}$$

The dimension of the space of solutions to (4.9) is then equal to the number of unbroken supersymmetries. The conventions for the gamma matrices are explained in the appendix. After choosing the physical gauge we can write (4.9) as

$$\sqrt{-g}\epsilon = \left(1 - \Gamma^i \partial_i X^A \Gamma_A - \frac{1}{2}\Gamma^{ij} \partial_i X^A \partial_j X^B \Gamma_{AB}\right) \Gamma_\star \epsilon \,, \tag{4.11}$$

where $\Gamma_\star = \Gamma_{012}$. Equating powers in ∂X, the zeroth order condition

$$\Gamma_\star \, \epsilon = \epsilon \tag{4.12}$$

can be rewritten using (A.2) and (A.3) as

$$\left(\gamma_{012} \otimes \tilde{\Gamma}_9\right) (\chi \otimes \eta) = (\chi \otimes \eta) \tag{4.13}$$

where we see that $\gamma_{012}\chi = \chi$, which is trivial, and that η is chiral in the eight-dimensional sense and has therefore eight independent real components. This condition defines the vacuum of the theory. As it happens, the chirality constraint on η is automatically satisfied by solutions to (4.4) as can be checked again by using the explicit representation mentioned above. Therefore, (4.4) are the conditions which determine the supersymmetry preserved by the vacuum of the theory, namely it preserves 3/16 of the 32 M theory supersymmetries.

JHEP03(2004)051

To first order equation (4.11) reads

$$\Gamma^i \partial_i X^A \Gamma_A \, \epsilon = 0 \,. \tag{4.14}$$

Before we study the implications of (4.14), we will show that this condition implies that the full equation (4.11) is satisfied. By iterating (4.14) we obtain the relations

$$\begin{aligned} \Gamma^{ij} \partial_i X^A \partial_j X^B \Gamma_{AB} \, \epsilon &= -\eta^{ij} \tilde{g}_{ij} \, \epsilon \\ \Gamma^{ijk} \partial_i X^A \partial_j X^B \partial_k X^C \Gamma_{ABC} \, \epsilon &= 0 \\ 2\eta^{ij} \eta^{kp} \tilde{g}_{ik} \tilde{g}_{jp} \, \epsilon &= (\eta^{ij} \tilde{g}_{ij})^2 \, \epsilon \,, \end{aligned} \tag{4.15}$$

where we have defined $\tilde{g}_{ij} = \partial_i X^I \partial_j X^J g_{IJ}$. These relations and the expansion

$$\det(-\tilde{g}) = 1 + \eta^{ij} \tilde{g}_{ij} + \frac{1}{2} (\eta^{ij} \tilde{g}_{ij})^2 - \frac{1}{2} \eta^{ij} \eta^{kp} \tilde{g}_{ik} \tilde{g}_{jp} \tag{4.16}$$

for the determinant appearing in (4.11) are all that are needed to establish that studying the condition (4.14) is sufficient to ensure supersymmetry. This is the statement that the equations defining a Kähler calibration are linear and first order in derivatives. This is not true for other calibrations [15], although equation (4.14) will still reproduce these to first order [4].

We can now proceed to understand the implications of (4.14). To do so, let us introduce a vierbein for the metric (3.4)

$$\begin{aligned} e^X &= U^{1/2} \left(dX + \frac{\lambda}{U} dY \right) \\ e^Y &= \left(\frac{UV - \lambda^2}{U} \right)^{1/2} dY \\ e^\varphi &= \left(\frac{V}{UV - \lambda^2} \right)^{1/2} \left(d\varphi - \frac{\lambda}{V} d\psi \right) \\ e^\psi &= V^{-1/2} d\psi \,, \end{aligned} \tag{4.17}$$

and use it to write the curved Dirac matrices in terms of the tangent space ones $\Gamma_A = (\Gamma_{X_3}, \Gamma_{Y_3}, \Gamma_\varphi, \Gamma_\psi)$, where we have introduced the subscript 3 to make a direct comparison with section 4.1. Using this notation and the new lump equations (3.8), the condition (4.14) implies the following two projectors

$$\begin{aligned} \left(\lambda \Gamma^2 \Gamma_\varphi + \Gamma^1 \sqrt{UV} \Gamma_{Y_3} - \Gamma^2 \sqrt{UV - \lambda^2} \Gamma_\psi \right) \epsilon &= 0 \\ \left(\lambda \Gamma^2 \Gamma_\psi - \Gamma^1 \sqrt{UV} \Gamma_{X_3} + \Gamma^2 \sqrt{UV - \lambda^2} \Gamma_\varphi \right) \epsilon &= 0 \,. \end{aligned} \tag{4.18}$$

We can write these succintly as

$$\begin{aligned} \left(\Gamma^{12} \Gamma_{Y_3 \psi} \cosh \tau + \Gamma_{\psi\varphi} \sinh \tau \right) \epsilon &= \epsilon \\ \left(\Gamma^{12} \Gamma_{X_3 \varphi} \cosh \tau + \Gamma_{\psi\varphi} \sinh \tau \right) \epsilon &= \epsilon \,, \end{aligned} \tag{4.19}$$

JHEP03(2004)051

where the parameter τ is defined by

$$\cosh\tau = \frac{\sqrt{UV}}{\sqrt{UV-\lambda^2}} \qquad \sinh\tau = \frac{\lambda}{\sqrt{UV-\lambda^2}}\,. \tag{4.20}$$

It is now possible to use the explicit representation (A.3) to impose the conditions (4.19) on the general form of the covariantly constant spinor (4.8). On doing this it is easy to check that only two of the arbitrary constants a_{si} remain independent, signifying that general solutions preserve two supersymmetries. We may ask ourselves whether a subset of the solutions to the equations (3.8), for example, whether setting $Y=0$ identically, may preserve more supersymmetry, but it can be checked that this is not the case.

It is possible to run through the same argument in the $\lambda=0$ case. In this case the equations satisfied by the lumps are (3.9) and the projector (4.14) imposes the conditions

$$\begin{aligned}\Gamma^{12}\Gamma_{Y_3\psi}\,\epsilon &= \epsilon\\ \Gamma^{12}\Gamma_{X_3\varphi}\,\epsilon &= \epsilon\,.\end{aligned} \tag{4.21}$$

These commute between themselves and with the other conditions imposed on the covariantly constant spinors in this case. We note that we can have solutions where only one of the pairs (X_3,φ) or (Y_3,ψ) is non-zero. This would be a solutions with two active fields. From the point of view of the sigma model, all solutions, with either two or four active fields, preserve four supersymmetries. This is not the case in the context of M theory, where only solutions with two active fields preserve four supersymmetries and solutions with four active fields preserve only two.

A summary of the results is presented in table 1 displayed in the Introduction.

5. Conclusions and discussions

We summarise the results of the paper as follows. We studied a $(2+1)$-dimensional sigma model with an HK_8 target space and derived the BPS equations that must be satisfied by lump solitons. The sigma model seemed to suggest that solutions to these equations preserve 4 supersymmetries. We then analysed the membranes as probes in the M theory background (4.1). We checked explicitly that the vacuum of the theory generically has less supersymmetries than the sigma model, and that solutions to the lump equations, which we expect to hold in M theory preserve only two supersymmetries.

The setup considered here of an M2-brane in the background (4.1) is dual to a Hanany-Witten configuration [5] in which the D3-brane suspended between an NS5-brane and a (p,q)-5-brane [17, 18] (the choice of $W=\lambda$ is constant is necessary for this duality). The effective field theory on the D3-brane is a $(2,1)$-dimensional gauge theory. When the $\mathrm{HK}_8=\mathrm{HK}_4\times\mathrm{HK}_4$ this dual theory will also have $\mathcal{N}=4$ supersymmetry, but for generic HK_8, that is, when $\lambda\neq 0$, this theory will include a Chern-Simons term that will preserve only $\mathcal{N}=3$. In fact, we expect the lumps studied above to correspond to vortices in such a model, and following the results in this paper, we expect such vortices to preserve $\frac{1}{3}$ of the 6 supercharges, in agreement with [19, 20].

JHEP03(2004)051

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A. Gamma matrix conventions

Throughout we take the 11-dimensional Dirac matrices to satisfy the Clifford algebra

$$\{\Gamma_M, \Gamma_N\} = 2\, G_{MN}\,. \tag{A.1}$$

Following the decomposition

$$\mathrm{SO}(10,1) \to \mathrm{SO}(2,1) \times \mathrm{SO}(8) \tag{A.2}$$

we can write these in tensor product form as

$$\Gamma_M = (\Gamma_i, \Gamma_A) = (\gamma_i \otimes \tilde{\Gamma}^9, 1 \otimes \tilde{\Gamma}_A)\,, \tag{A.3}$$

where $M = 0, \ldots, 9, \natural$ ($\natural$ denotes the 11-th dimension), $i = 0, 1, 2$ are the worldvolume directions and $A = 3, \ldots, 9, \natural$ the HK_8 directions. The γ_i are a representation of the $\mathrm{SO}(2,1)$ matrices given explicitly by

$$\gamma_0 = \begin{pmatrix} 0 & 1 \\ -1 & 0 \end{pmatrix} \qquad \gamma_1 = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix} \qquad \gamma_2 = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}, \tag{A.4}$$

and the $\tilde{\Gamma}$ are a representation of the SO(8) gamma matrices. We take this to be the representation described in an appendix to section 5 of [16] which is real. $\tilde{\Gamma}^9$ is the chirality matrix in eight dimensions, defined by $\tilde{\Gamma}^9 = \tilde{\Gamma}_{[3 \ldots} \tilde{\Gamma}_{\natural]}$ and satisfying $(\tilde{\Gamma}^9)^2 = 1$ and $\{\tilde{\Gamma}^9, \tilde{\Gamma}_A\} = 0$. This ensures that the product of all the 11 dimensional Dirac matrices is the identity.

Following (A.2) we can write an 11-dimensional spinor ϵ as

$$\epsilon = \chi(\sigma^i) \otimes \eta(X^A)\,. \tag{A.5}$$

The charge conjugation matrix in the 11-dimensional representation (A.3) is given by Γ_0. As usual the spinor ϵ satisfies a Majorana condition in 11 dimensions which requires it to be real and therefore have 32 independent real components.

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JHEP03(2004)051